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AN APPROXIMATION OF THE FORM OF A COMPRESSED FLEXIBLE ROD

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A compact algorithm is proposed for exact calculation of the coordinates of the plane elastic line of an axially compressed flexible rod under any loads. Refined approximate formulas are obtained for calculation of the coordinates of the elastic line with an error not greater than 1% *of the rod length even for loads which exceed the critical Euler load by* 30%.

The postcritical forms of equilibrium of compressed flexible rods were investigated for the first time by Lagrange and studied in detail with the use of the tables of elliptic integrals by Krylov [1] and Popov [2]. However, the methods given in [1, 2] are laborious for practice because of the need to calculate the quantities expressed in elliptic integrals of the first and second kinds.

Formulation of the Problem. Exact Solution. We consider a flexible, elastic, simply supported rod which is compressed by the axial force P whose magnitude and direction do not change during deformation of the rod. We assume that the length L of the axial line of the rod remains unchanged and the bending of the line occurs only in the (x, y) plane. We analyze the stable form of equilibrium of the rod under the load $P > P_* = EI(\pi/L)^2$ (EI is the flexural rigidity of the rod), i.e., under a load that exceeds the first Euler critical load [1–4].

We introduce the notation for the incomplete and complete elliptic integrals of the first and second kinds:

$$F(\varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad F = F\Big(\frac{\pi}{2}\Big), \qquad E(\varphi) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \quad E = E\Big(\frac{\pi}{2}\Big).$$

Let $\lambda = P/P_*$ be the dimensionless load parameter (the compressive load normalized to the Euler critical load), a and $f = a/L$ be the maximum dimensional and dimensionless deflections of the rod, $t = s/L$ be the distance from the end of the rod to the point at the axis divided by the length of the rod, and $x(t)$ and $y(t)$ be the desired dimensionless Cartesian coordinates (normalized to the length L) of the point t ($0 \leqslant t \leqslant 1$). To calculate with any accuracy the coordinates of the point which lies on the axis of the rod bent by the axial load λ at the distance s ($0 \leqslant s \leqslant L$) from its end (the distance is measured along the axis), the following algorithm [1–4] is proposed:

Step 1. For a given load $\lambda \geqslant 1$, calculate the parameter k ($0 \leqslant k < 1$) from the equation

$$F = \pi\sqrt{\lambda}/2.$$

The parameter k is geometrically interpreted as the sine of the half-angle between the tangent to the axis of the bent rod at its end and the initially rectilinear axis of the rod. For example, $k = 0$ corresponds to $\lambda = 1$ and $k \to 1$ as $\lambda \to \infty$.

Step 2. Calculate the maximum deflection which occurs for $t = 1/2$, i.e., in the middle of the rod: $f = 2k/(\pi\sqrt{\lambda})$.

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Step 3. For a given value of $t = s/L$, find the amplitude φ $(-\pi/2 \leqslant \varphi \leqslant \pi/2)$ of the elliptic integral from the equation

$$F(\varphi) = (1/2 - t)\pi\sqrt{\lambda}.$$

Step 4. Calculate the coordinate $y(t) = f\cos\varphi$.

Step 5. Calculate the coordinate $x(t) = 2(E - E(\varphi))/(\pi\sqrt{\lambda}) - t$.

We note that only the complete elliptic integral of the first kind is used to calculate the maximum deflection f (steps 1 and 2). Many approximate formulas for calculation of the maximum deflection are available, the most exact of which

$$f = (4/(\sqrt{3}\,\pi\lambda))\sqrt{\sqrt{6\lambda - 2} - 2} \tag{1}$$

is given in [5]. However, to calculate the coordinate $y(t)$ in the general case, it is necessary to solve an equation (step 3), in which the desired φ is the limit of an incomplete elliptic integral of the first kind. Moreover, to calculate the coordinate $x(t)$, the values of elliptic integrals of the second kind are required (step 5).

Approximate Formulas. There are very few approximate formulas for calculation of the coordinates $x(t)$ and $y(t)$. One of the known formulas is

$$y(t) = (2\sqrt{2}/\pi)\sqrt{\lambda - 1}\sin\pi t \tag{2}$$

(see [6]) which is the particular case of the formula $y(t) = c\sin\pi t + c_1\sin 3\pi t$ given therein with a wrong $(cc_1 \geqslant 0)$ value of the coefficient c_1. The formula obtained by the perturbation method

$$y(t) = \alpha\sin\pi t - (\pi^2/64)\alpha^3\sin 3\pi t \tag{3}$$

is more exact [7, 8]. Here α is found from the relation $f^2 = \alpha^2(1 + (\alpha\pi)^2/32)$ in terms of f or directly in terms of λ using (1). The exact solution and the results of calculations using formulas (2) and (3), the equation

$$y(t) = \alpha\sin\pi t, \tag{4}$$

which is the truncation of (3), and the equation

$$y(t) = f\sin\pi t \tag{5}$$

are given in Table 1 for $\lambda \approx 1.2939$ $(k = \sin 40°)$ and various t. For $\lambda \approx 1$, one can obtain from (5) the formula $y(t) = (2\sqrt{2}/(\pi\lambda))\sqrt{\lambda - 1}\sin\pi t$, which is more exact than (2) but less exact than formulas (3)–(5); therefore, the corresponding calculation results are not given in Table 1. Since, for a parametrically specified curve, we have

$$x(t) = \int_0^t \sqrt{1 - (y'(t))^2}\,dt = \int_0^t \Big[1 - \frac{1}{2}(y'(t))^2 - \frac{1}{8}(y'(t))^4 - \ldots\Big]dt,$$

for $|y'(t)| < 1$, using (5), one can approximately represent the coordinate $x(t)$ in the form

$$x(t) \approx \Big[1 - \frac{(f\pi)^2}{4} - \frac{3(f\pi)^4}{64} - \frac{5(f\pi)^6}{256}\Big]t - \frac{4 + (f\pi)^2}{32\pi}(f\pi)^2\sin 2\pi t. \tag{6}$$

The calculation results for the coordinate $x(t)$ by formula (6), which provides the best accuracy compared to the other formulas tested, are given in Table 1.

Comparison of the Formulas. Conclusions. An analysis of the approximate formulas for determination of the form of the bent elastic rod shows that formulas (3) and (6) provide the best accuracy in calculating the coordinates $y(t)$ and $x(t)$ for a specified t. For loads of up to $\lambda = 1.3$, the calculation errors for $y(t)$ and $x(t)$ are less than 1 and 0.3%, respectively, relative to the length of the rod L (see Table 1). Formulas (3)–(5) are significantly more exact than formula (2). Calculation of the coordinate $y(t)$ by formula (2) gives an error which attains 13% for the loads considered.

For $\lambda \approx 1.2939$, Fig. 1 shows the forms of the elastic curves constructed by the exact formulas of steps 1–5 (curve 1), approximate formulas (2) and (6) (curve 2), and formulas (5) and (6) (curve 3). Curve 3 lies

TABLE 1

Coordinates $x(t)$ and $y(t)$ of the Rod for $\lambda = 1.2939$

Coordinate	Formula	t				
		0.1	0.2	0.3	0.4	0.5
$y(t)$	(2)	0.1508	0.287	0.395	0.464	0.488
	(3)	0.1036	0.201	0.283	0.340	0.360
	(4)	0.1091	0.207	0.286	0.334	0.353
	(5)	0.1111	0.211	0.291	0.342	0.360
	Exact solution	0.0981	0.193	0.277	0.337	0.360
$x(t)$	(6)	0.0170	0.049	0.105	0.186	0.282
	Exact solution	0.0194	0.051	0.104	0.183	0.280

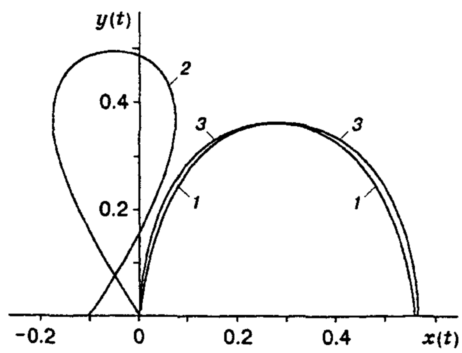

Fig. 1

above curve 1 everywhere. In the calculation of the coordinate $x(t)$ for curve 2, the value of f in formula (6) was set equal to the maximum of the function (2), i.e., it was assumed that $f = 2\sqrt{2\lambda - 2}/\pi$. If formulas (3) and (6) are used to determine the elastic line, the corresponding curve lies between curves 1 and 3, i.e., it is closer to the exact curve 1 than curve 3. However, bearing in mind that formulas (3) and (4) are more complex compared with formula (5) and the refinement of the results is insignificant, the use of formula (5) is advantageous.

Formulas (1), (5), and (6) are expedient in preliminary calculations in more complicated problems where rods are the elements of a structure. These formulas can be useful in instrument making, in particular, for calculating mechanical regulators.

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